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Technical Report No. 32-431 (Part II)

A Simplified Statistical Model for Missile Launching—II

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CONTENTS

I. Introduction 1

II. The Problem 2

III. The Principal Results—Exact Expressions 3

IV. Approximate Expressions 5

V. Derivation of Principal Results 5

TABLES

Table 1. A comparison of the mean and median number of days 5

FIGURES

1. Cumulative probability of launching three vehicles from two pads, $R \geq 18$ 4

2. Cumulative probability of launching three vehicles from two pads, $R = 1$ 4

ABSTRACT

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This Report is the second of a series dealing with the problem of estimating the number of days necessary to complete the countdown procedures for the launching of three vehicles from two launch pads. The first report dealt with the situation permitting simultaneous countdowns to occur, although simultaneous launchings were disallowed. In the present Report, a simultaneous countdown is permitted only between the second and third vehicles, although again, no simultaneous launchings are allowed. The model for the delays encountered in both cases remains unchanged.

I. INTRODUCTION

In a previous report,* it was pointed out that in the preliminary planning of any space mission feasibility study it is necessary to know the estimated number of days it will take to launch a given number of vehicles from a given number of launch pads under various assumptions concerning the manner in which the countdown procedures are conducted, and the nature of the delays encountered.

In that report, an analysis was made of the specific problem of launching three vehicles from two launch pads when simultaneous countdowns could be conducted on the erected vehicles, although simultaneous launches were disallowed. In the present Report, a modified procedure is considered. Simultaneous launches are still not permitted at any time (for obvious practical reasons) nor is a simultaneous countdown on the two initially erected vehicles. However, a simultaneous countdown is allowed to occur between the second and third vehicles if the former still has not been launched by the time the third has been erected. Practical situations can be envisioned in which such a mode of operation is not only feasible but reasonable.

Both reports have preserved an extremely simplified concept of the delays expected to be encountered. Future reports will deal with the same countdown procedures considered here but will generalize the model for the nature of the delays to a much more realistic (and necessarily complex) situation. In addition, reports are anticipated which will encompass even more sophisticated models of both the countdown procedure and delay models.

*Solloway, C. B., *A Simplified Statistical Model for Missile Launching—I,* Technical Report No. 32-431 (Part I), Jet Propulsion Laboratory, Pasadena, May 1, 1963.

II. THE PROBLEM

In this Report, we are concerned with the number of days necessary to complete the countdown procedures for the launching of three vehicles from two pads under the following assumptions:

1. Two vehicles are erected simultaneously on the pads, and the countdown proceeds on one vehicle.
2. When the countdown has been successfully completed on the first vehicle, the countdown is initiated on the second vehicle the following day.
3. Simultaneously, the vacated pad is immediately cleaned and prepared for the third vehicle. There is a (fixed) period of R days' delay after a launching before the same pad may be utilized for a second launch attempt (the turnaround time).
4. As soon as the third vehicle has been erected on the vacated pad, the countdown procedure begins, simultaneously with that of the second vehicle if that countdown has not yet been successfully completed.
5. The second and third vehicles may not be launched on the same day. If one countdown is completed, the other is terminated and not resumed until the following day.
6. Each vehicle is independent of, and identical to, the others. On any single countdown attempt, there is a probability p of a successful completion and a probability $q = 1 - p$ of failure. Any failure results in the termination of that countdown attempt, and a new attempt is made *the following day*. That is, any failure leads to a one-day delay. It is assumed that a successful countdown attempt can be completed in one day.
7. The failure to complete a countdown does not affect the subsequent attempts in any way. That is, the trials are independent from day to day as well as from vehicle to vehicle.

Fundamentally, the model discussed here differs from that in the first report only in that a simultaneous countdown is not permitted between the two vehicles originally on the launch pads.

III. THE PRINCIPAL RESULTS—EXACT EXPRESSIONS

Let N be the number of days until the third successful countdown. Then, the exact frequency function for N is given by

$$f(N) = \text{probability of completing the third countdown on the } N\text{th day}$$

$$= \left(1 - \frac{pq^R}{1+q}\right)^{-1} \times \left\{ \begin{aligned} & p^2 q^{N-R-2}\left[(N-R-1) - \frac{q^R - q^{N-1}}{p}\right] \\ & + p^2 q^{N-2}\left[(N-R-2) - \frac{q(1-q^{N-R-2})}{p}\right] \end{aligned} \right\}$$

$$N \geq R+2 \qquad (1)$$

The cumulative distribution function for N is given by

$$F(N) = \text{probability of completing the third countdown on or before the } N\text{th day}$$

$$= \sum_{x=R+2}^{N} f(x)$$

$$= \left(1 - \frac{pq^R}{1+q}\right)^{-1} \times \left\{ \begin{aligned} & [1-(N-R)\,pq^{N-R-1} - q^{N-R}] \\ & + q^{R+1}\,[1-(N-R-1)\,pq^{N-R-2} - q^{N-R-1}] \\ & - q^R(1+q)\,(1-q^{N-R-1}) + \frac{2q^{R+1}}{1+q}\left[1 - q^{2(N-R-1)}\right] \end{aligned} \right\} \qquad (2)$$

and the moment generating function $M(\theta)$ for N is given by

$$M(\theta) = \sum_{x=R+2}^{\infty} e^{\theta x} f(x)$$

$$= \left(1 - \frac{pq^R}{1+q}\right)^{-1} e^{\theta(R+2)} \left[p^2 \frac{1}{(1-e^\theta q)^2} - pq^R \left(\frac{1}{1-e^\theta q} - \frac{q}{1-e^\theta q^2}\right) + \frac{p^2 q^{R+1} e^\theta}{(1-e^\theta q)^2} - pq^{R+1}\left(\frac{1}{1-e^\theta q} - \frac{1}{1-e^\theta q^2}\right)\right] \qquad (3)$$

from which we obtain the mean μ_N and variance σ_N^2 in the usual manner. Thus,

$$\mu_N = \left.\frac{dM(\theta)}{d\theta}\right|_{\theta=0}$$

$$= (R+2) + \left(1 - \frac{pq^R}{1+q}\right)^{-1} \left[\frac{2q}{p} + \frac{2q^{R+3}}{p(1+q)}\right]$$

$$\sigma_N^2 = \left.\frac{d^2M}{d\theta^2}\right|_{\theta=0} - \mu_N^2 \qquad (4)$$

The variance is quite complicated; its approximation is given in Section IV.

Figures 1 and 2 show the exact cumulative distribution function $F(N)$ for $p = 0.2, 0.3$, and 0.4 for $R \geq 18$ and $=1$, respectively. For large R (≥ 18) the curves are indistinguishable as a function of R, so they are plotted as a function of $N - R$.

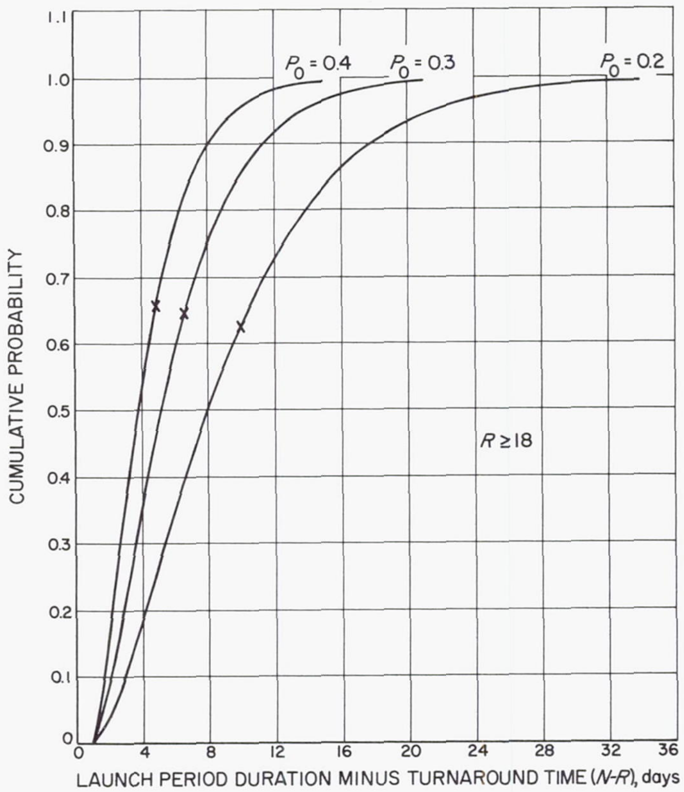

Fig. 1. Cumulative probability of launching three vehicles from two pads, $R \geq 18$

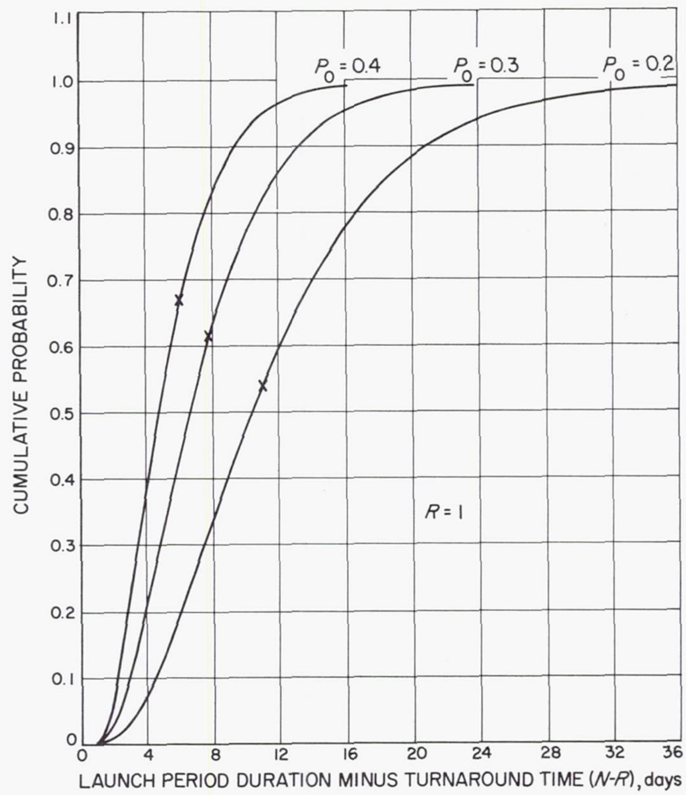

Fig. 2. Cumulative probability of launching three vehicles from two pads, $R = 1$

IV. APPROXIMATE EXPRESSIONS

The expressions in Eq. (1) to (4) are rather complicated. In most practical situations, R is quite large (e.g., an optimistically small realistic R today is about 18 days), and even with a small probability of success p, major simplifications can be obtained. The realism of the model does not justify the accuracy necessary to include these terms. Making the approximations, we obtain

$$f(N) \simeq p^2 q^{N-R-2}(N-R-1) \tag{5}$$

$$F(N) \simeq 1 - (N-R)\,pq^{N-R-1} - q^{N-R} \tag{6}$$

$$M(\theta) \simeq \frac{p^2\, e^{\theta(R+2)}}{(1-e^{\theta}q)^2} \tag{7}$$

$$\mu_N \simeq (R+2) + \frac{2q}{p} \tag{8}$$

$$\sigma_N^2 \simeq \frac{2q}{p^2} \tag{9}$$

We also note in passing that for large R (≥ 18), the mean number of days to launch the three vehicles (μ_N) is approximately equal to the median number of days. (The median number of days is such that the probability is 0.5 that the three vehicles will be launched on or before that day.) This number is given as the abscissa of the curve for which $F(N) = 0.5$ (Fig. 1 and 2). The approximation is not too poor even for small R. Thus, for example, we obtain Table 1 from Eq. (8) and Fig. 1 and 2.

Table 1. A comparison of the mean and median number of days

R	p	0.2	0.3	0.4
18	μ_N	28	25	23
	Median	26	25	24⁻
1	μ_N	11	8⁻	6
	Median	10⁺	7⁻	5⁻

V. DERIVATION OF PRINCIPAL RESULTS

Under the assumptions of Section II, the probability of the first successful countdown on the kth trial for any vehicle is clearly

$$p_k = pq^{k-1} \qquad k = 1,2,\ldots \tag{10}$$

(i.e., the Pascal or geometric distribution). Suppose now that we label the pads 1 and 2 and the first vehicle on each pad by the same number, the standby vehicle being labeled 3. From the conditions stated, vehicle 3 must go from pad 1. Now, there are two distinct and mutually exclusive ways in which the countdowns can be successful.

Case	*Order of Successful Countdowns* (Vehicle No.)
1	1-2-3
2	1-3-2

The probability of three successful countdowns in exactly N days, given case 1, is

$$P\{N \text{ days} \mid \text{case } 1\} = \sum_{k=1}^{N-1-R} \sum_{m=1}^{N-1-k} P\left\{\begin{array}{l}\text{No. 1 took } k \text{ trials,}\\ \text{No. 2 took } m \text{ trials, and}\\ \text{No. 3 took } N-R-k \text{ trials}\end{array}\right\} \tag{11}$$

where the limits for m and k are easily understood from the following sketch:

$\longleftarrow$ N days $\longrightarrow$

k days	R days	$N-k-R$ days
No. 1	turn-around	No. 3

$1 \leq k \leq N-R-1$ trials at most available to No. 1

$\longleftarrow$ N days $\longrightarrow$

k days	$N-1-k$ days	1 day
No. 1	No. 2	No. 3

$1 \leq m \leq N-1-k$ trials at most available to No. 2

From the assumed independence of the vehicles, the joint event appearing in the summation of Eq. (11) can be written in the form

$$
\begin{aligned}
P\{N \text{ days} \mid \text{case } 1\} &= \sum_{k=1}^{N-1-R} \sum_{m=1}^{N-1-k} p_k\, p_m\, p_{N-R-k} \\
&= \sum_{k=1}^{N-1-R} p^2 q^{k-1+N-R-k-1} \sum_{m=1}^{N-1-k} p\, q^{m-1} \\
&= \sum_{k=1}^{N-1-R} p^2 q^{N-R-2} (1 - q^{N-1-k}) \\
&= p^2 q^{N-R-2} \left[(N-R-1) - \frac{q^{N-2} - q^{N-1-(N-R)}}{1 - \dfrac{1}{q}} \right] \\
&= p^2 q^{N-R-2} \left[(N-R-1) - \frac{q^R - q^{N-1}}{p} \right]
\end{aligned} \tag{12}
$$

Similarly, for case 2, we have

$$
P\{N \text{ days} \mid \text{case } 2\} = \sum_{k=1}^{N-R-2} \sum_{m=1}^{N-R-k-1} P \left\{ \begin{array}{l} \text{No. 1 took } k \text{ trials,} \\ \text{No. 2 took } N-k \text{ trials, and} \\ \text{No. 3 took } m \text{ trials} \end{array} \right\} \tag{13}
$$

where, as before, the limits for m and k are easily deduced from a consideration of the following sketch:

$\longleftarrow$ N days $\longrightarrow$

k days	R days	m days	1 day
No. 1	turn-around	No. 3	No. 2

$1 \leq m \leq N-1-R-k$ trials at most available to No. 3

$\longleftarrow$ N days $\longrightarrow$

k days	R days	1 day	1 day
No. 1	turn-around	No. 3	No. 2

$1 \leq k \leq N-R-2$ trials at most available to No. 1

Thus,

$P\,\{N \text{ days} \mid \text{case } 2\}$

$$= \sum_{k=1}^{N-R-2} \sum_{m=1}^{N-R-k-1} p_k\, p_{N-k}\, p_m$$

$$= \sum_{k=1}^{N-R-2} p^2\, q^{k-1+N-k-1} \sum_{m=1}^{N-R-k-1} p\, q^{m-1}$$

$$= \sum_{k=1}^{N-R-2} p^2\, q^{N-2}\,(1 - q^{N-R-k-1})$$

$$= p^2\, q^{N-2}\left[(N - R - 2) - \frac{q^{N-R-2} - q^{N-R-1-(N-R-1)}}{1 - \dfrac{1}{q}}\right]$$

$$= p^2\, q^{N-2}\left[(N - R - 2) - \frac{q(1 - q^{N-R-2})}{p}\right] \tag{14}$$

The sum of Eq. (12) and (14) would be the answer to the problem if simultaneous launchings were allowed but not counted. Since they have been disallowed (in the sample space of the experiment), we must renormalize the sample space. To do this, we ask what the probability of a simultaneous launch is when countdowns proceed simultaneously. This question is easily answered, for a simultaneous launch can occur in exactly N days in only one way; i.e., that vehicles No. 2 and No. 3 go together. The probability of this occurrence is

$$P_1\,\{N \text{ days}\} = \sum_{k=1}^{N-1-R} P \left\{ \begin{array}{l} \text{No. 1 took } k \text{ trials,} \\ \text{No. 2 took } N-k \text{ trials,} \\ \text{No. 3 took } N-R-k \text{ trials} \end{array} \right\} \tag{15}$$

where the appropriate sketch is

$\longleftarrow$ N days $\longrightarrow$

k days	R days	$N-k-R$ days
No. 1	turn-around	No. 3

No. 2 $\longrightarrow$

$\longleftarrow$ N days $\longrightarrow$

k days	R days	1 day
No. 1	turn-around	No. 2, 3

$1 \le k \le N - R - 1$ trials at most available to No. 1

Thus,

$$P_1\,\{N \text{ days}\} = \sum_{k=1}^{N-1-R} p_k\, p_{N-k}\, p_{N-R-k}$$

$$= \sum_{k=1}^{N-1-R} p^3\, q^{k-1+N-k-1+N-R-k-1}$$

$$= \sum_{k=1}^{N-1-R} p^3\, q^{2N-R-k-3}$$

$$= p^3 \left[\frac{q^{2N-R-4} - q^{2N-R-3-(N-R)}}{1 - \dfrac{1}{q}}\right]$$

$$= p^2\,(q^{N-2} - q^{2N-R-3})$$

$$= p^2\, q^{N-2}\,(1 - q^{N-R-1}) \tag{16}$$

The total probability of a simultaneous launching is

$$\sum_{N=R+2}^{\infty} P_1\,\{N\} = \sum_{N=R+2}^{\infty} p^2\, q^{N-2}\,(1 - q^{N-R-1})$$

$$= p^2 \left(\frac{q^R}{1-q} - \frac{q^{R+1}}{1-q^2}\right)$$

$$= p^2\, q^R \left[\frac{1}{p} - \frac{q}{p\,(1+q)}\right]$$

$$= \frac{p\, q^R}{1+q} \tag{17}$$

and the appropriate normalizing factor to apply is

$$\left(1 - \sum_{N=R+2}^{\infty} P_1\,\{N\}\right)^{-1} = \left(1 - \frac{p\, q^R}{1+q}\right)^{-1} \tag{18}$$

It is also easily verified that if we sum the right-hand sides of Eq. (12) and (14), we obtain $[1 - p\, q^R/(1+q)]$, thus verifying the appropriateness and correctness of this normalizing factor.

The derivation of the moment-generating function is straightforward and involves only the summing of series of the type encountered above. The details are left to the interested reader, as are the details of obtaining the mean and variance from $M(\theta)$.

The approximation formulas of Section IV are obtained from the exact formulas of Section III by deleting all terms with a factor of q^R or q^N (but not terms like q^{N-R}). Physically, this is equivalent to the rather obvious fact that for large R, case 2 of Section III is a second-order effect. That is, the probability of launching two vehicles from a single pad before the other one gets away is very small.